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# Single-Frame Pose Estimation and Performance Analysis

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*ABSTRACT*

**This study presents an error analysis of a specific single-frame dual-quaternion batch estimator that employs point and unit vector measurements. The estimator addresses a constrained least-squares optimization by minimizing a cost function that aggregates orientation and position errors. An eigenvalue-based error analysis is developed for the first time for that dual quaternion estimator, yielding closed-form expressions for the three-dimensional attitude and translation error vectors and their covariance matrices. Extensive Monte Carlo simulations validate the accuracy of these analytical predictions.**

**Keywords:** Batch Pose Estimation, Dual Quaternions, Error Analysis

## Nomenclature

| | | |
|---|---|---|
| $\boldsymbol{\mathcal{q}}$ | = | dual quaternions |
| $\boldsymbol{q}$ | = | real part of a dual quaternion |
| $\boldsymbol{q}_d$ | = | dual part of a dual quaternion |
| $\boldsymbol{\delta\theta}$ | = | 3D rotation error vector |
| $\boldsymbol{\Delta t}$ | = | 3D translation error vector |
| $P_{\delta\theta}$ | = | attitude error covariance matrix |
| $P_{\Delta t}$ | = | translation error covariance matrix |

## 1 Introduction

Accurate pose estimation is fundamental in aerospace applications, particularly for spacecraft proximity operations, rendezvous, and docking. Among various pose estimation approaches, dual-quaternion-based methods offer computational advantages by unifying rotation and translation within a single algebraic framework. Dual quaternions combine Hamilton's quaternion theory in [1] and Clifford's dual number theory in [2] to represent rigid-body pose. Compared to transformation matrices, dual quaternions reduce computational load and storage demands. An efficient dual-quaternion estimator was introduced in [3] that simultaneously estimates orientation and position by minimizing a cost function that aggregates both error types. However, despite its widespread adoption, no rigorous error analysis of this estimator has been developed to date. In particular, closed-form expressions for the covariance matrices of attitude

and translation estimation errors remain unavailable, which are essential for assessing estimation accuracy and integrating with navigation filters. While error analysis for attitude-only estimators is well established in [4], the coupled pose estimation case presents unique challenges due to the interplay between rotation and translation errors through the dual-quaternion structure. The main contribution of this work is the first derivation of analytical expressions for the covariance matrices of a dual quaternion batch estimator in [3]. Specifically, we develop first-order expressions for the three-dimensional attitude error vector and translation error vector. The closed-form covariance matrix formulas enable practitioners to assess pose estimation accuracy without the computational expense of Monte Carlo simulations.
The paper is organized as follows: Section 2 reviews the fundamentals of dual quaternions and summarizes the estimator algorithm. Section 3 develops the error analysis, deriving expressions for error vectors and covariance matrices. Section 4 presents simulation results that validate the analytical expressions. Section 5 concludes the paper.

# 2 Summary of the dual quaternion estimator

A dual quaternion $\underset{\sim}{q}$ is defined as: $\underset{\sim}{q} = \boldsymbol{q} + \boldsymbol{q_d}$ where $\boldsymbol{q}$ and $\boldsymbol{q_d}$ are the real part and dual part, both being quaternions. The unit dual quaternion adheres to two constraints:

$$\boldsymbol{q}^T\boldsymbol{q} = 1 \tag{1}$$

$$\boldsymbol{q}_d^T\boldsymbol{q} = 0 \tag{2}$$

These two constraints ensure that dual quaternions can unambiguously represent rigid body transformations in three-dimensional space, with the degrees of freedom effectively reduced to six.
Given batches of point quaternion observations, $\{\tilde{\boldsymbol{p}}_{b_i}, \boldsymbol{p}_{r_i}\}_{i=1}^{l}$, and unit vector quaternion observations, $\{\tilde{\boldsymbol{n}}_{b_i}, \boldsymbol{n}_{r_i}\}_{i=1}^{k}$, compute the following four-dimensional matrices:

$$C_1 = -2\sum_{i=1}^{k}\alpha_i Q(\tilde{\boldsymbol{n}}_{b_i})^T W(\boldsymbol{n}_{r_i}) - 2\sum_{i=1}^{l}\beta_i Q(\tilde{\boldsymbol{p}}_{b_i})^T W(\boldsymbol{p}_{r_i}) \tag{3}$$

$$C_2 = \left(\sum_{i=1}^{l}\beta_i\right) I \tag{4}$$

$$C_3 = 2\sum_{i=1}^{l}\beta_i (W(\boldsymbol{p}_{r_i}) - Q(\tilde{\boldsymbol{p}}_{b_i})) \tag{5}$$

The operators $Q(\boldsymbol{q})$ and $W(\boldsymbol{q})$ are $4 \times 4$ matrix functions of the rotation quaternion $\boldsymbol{q}$ corresponding to left- and right-quaternion multiplication, respectively. They are expressed as follows:

$$Q(\boldsymbol{q}) = \begin{bmatrix} [\,\mathbf{e}\times] + qI_3 & \mathbf{e} \\ -\mathbf{e}^T & q \end{bmatrix} \tag{6}$$

$$W(\boldsymbol{q}) = \begin{bmatrix} -[\,\mathbf{e}\times] + qI_3 & \mathbf{e} \\ -\mathbf{e}^T & q \end{bmatrix} \tag{7}$$

In particular, the parameters $\boldsymbol{n}$ and $\boldsymbol{p}$ are vectors and points expressed as $4 \times 1$ pure quaternions in the body and reference frames. And the matrix $A$ as follows:

$$A = \frac{1}{2}(C_3(C_2 + C_2{}^T)C_3{}^T - C_1 - C_1{}^T) \tag{8}$$

Then, extract the optimal rotation quaternion estimate $\boldsymbol{q}$ as the eigenvector of $A$ associated with the maximal eigenvalue, and compute the dual part $\boldsymbol{q_d}$ as follows:

$$\boldsymbol{q_d} = -(C_2 + C_2{}^T)^{-1} C_3 \boldsymbol{q} \tag{9}$$

# 3 Error Analysis

This section conducts an error analysis of the algorithm that was previously outlined. We first derive expressions for the rotation error vector $\boldsymbol{\delta\theta}$ and the translation error vector $\mathbf{\Delta t}$. Then, we perform a random error analysis to ascertain the covariance matrix associated with these error vectors.

## 3.1 Pose Error Vector

Measurement noise significantly affects the accuracy of the data in both the reference frame and the body frame of the spacecraft, influencing the position and orientation calculations. We parameterize the attitude error using a rotation vector $\boldsymbol{\delta\theta}$ and the position error using a translation vector $\mathbf{\Delta t}$. These vectors represent the discrepancies in rotation and translation between the estimated body frame $\hat{\mathcal{B}}$ and the true body frame $\mathcal{B}$.

$$\hat{q} = q_{RB} \odot q_{B\hat{B}} = (\boldsymbol{q}^t + \epsilon \boldsymbol{q_d}^t) \odot (\delta \boldsymbol{q} + \epsilon \delta \boldsymbol{q_d}) = (\boldsymbol{q} \odot \delta \boldsymbol{q}) + \epsilon(\boldsymbol{q_d} \odot \delta \boldsymbol{q} + \boldsymbol{q} \odot \delta \boldsymbol{q_d}) \tag{10}$$

where $\odot$ means dual quaternion multiplication and quaternion multiplication. The error analysis utilizes first-order approximations in measurement errors. Equation. 10 can be approximated using Taylor series expansions as follows:

$$\hat{q} = (\boldsymbol{q}^t + \boldsymbol{q}^t \odot \begin{bmatrix} \frac{1}{2}\boldsymbol{\delta\theta} \\ 0 \end{bmatrix}) + \epsilon(\boldsymbol{q_d}^t + \boldsymbol{q_d}^t \odot \begin{bmatrix} \frac{1}{2}\boldsymbol{\delta\theta} \\ 0 \end{bmatrix} + \boldsymbol{q}^t \odot \begin{bmatrix} \frac{1}{2}\mathbf{\Delta t} \\ 0 \end{bmatrix} \odot \boldsymbol{q}^t) \tag{11}$$

where superscript $^t$ denotes the true value of the underlying variable. We are only considering the multiplicative error of the dual quaternion and not the additive error, and given that the additive noise of unit vectors and points is very small, these two points allow us to assume that the reference frame $\mathcal{R}$ and body frame $\mathcal{B}$ coincide. With this assumption, the true values of the dual quaternion can be simplified and Eq. 11 yields:

$$\hat{q} = (\begin{bmatrix} \mathbf{0_3} \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \boldsymbol{\delta\theta} \\ 0 \end{bmatrix}) + \epsilon(\frac{1}{2}\begin{bmatrix} \mathbf{\Delta t} \\ 0 \end{bmatrix}) \tag{12}$$

### *3.1.1 Attitude Error Vector*

First, we examine the attitude error. In our analysis, the vectors representing unit vectors and points in the body frame are denoted by $\mathbf{n_{b_i}}$ and $\mathbf{p_{b_i}}$, respectively, while those in the reference frame are represented by $\mathbf{n_{r_i}}$ and $\mathbf{p_{r_i}}$. These vectors comprise their true values, which are error-free, plus three-dimensional additive errors. Notably, when considering the reference frame as identical to the body frame, the true values of the reference and body vectors are equivalent. This is expressed mathematically as follows:

$$\mathbf{n_{b_i}} = \mathbf{n^t_{b_i}} + \mathbf{\Delta n_{b_i}} \tag{13}$$
$$\mathbf{n_{r_i}} = \mathbf{n^t_{b_i}} + \mathbf{\Delta n_{r_i}} \tag{14}$$
$$\mathbf{p_{b_i}} = \mathbf{p^t_{b_i}} + \mathbf{\Delta p_{b_i}} \tag{15}$$
$$\mathbf{p_{r_i}} = \mathbf{p^t_{b_i}} + \mathbf{\Delta p_{r_i}} \tag{16}$$

In this case, the attitude profile matrix $A$ can be rewritten:

$$A = \frac{1}{2}\left[(C_3^{0^T} + \Delta C_3{}^T)(C_2^0 + \Delta C_2)(C_3^0 + \Delta C_3) - C_1^0 - \Delta C_1 - C_1^{0^T} - \Delta C_1{}^T\right] = A_0 + \Delta A \tag{17}$$

The error-free matrix $A_0$ and the perturbation matrix $\Delta A$ are defined as follows:

$$A_0 = 4\begin{bmatrix} B_{0n} + B_{0p} - 2B_0 & \mathbf{0_{3\times1}} \\ \mathbf{0_{1\times3}} & k + \sum_{i=1}^{l} \mathbf{p^t_{b_i}}{}^{\mathrm{T}} \mathbf{p^t_{b_i}} \end{bmatrix} \tag{18}$$

$$\Delta A = 2\begin{bmatrix} (\Delta B_{0n} + \Delta B_{0p} - 2\Delta B) + (\Delta B_{0n} + \Delta B_{0p} - 2\Delta B)^T & \Delta Z \\ \Delta Z^T & 0 \end{bmatrix} \tag{19}$$

where

$$B_0 = \frac{1}{l}\sum_{i,j}^{l}\left[[\mathbf{p^t_{b_i}}\times][\mathbf{p^t{}_{b_j}}\times]\right]$$

$$B_{0n} = \sum_{i=1}^{k}\left[[\mathbf{n^t_{b_i}}\times][\mathbf{n^t_{b_i}}\times] + \mathbf{n^t_{b_i}}\mathbf{n^t_{b_i}}{}^{\mathbf{T}}\right]$$

$$B_{0p} = \sum_{i=1}^{l}\left[[\mathbf{p^t_{b_i}}\times][\mathbf{p^t_{b_i}}\times] + \mathbf{p^t_{b_i}}\mathbf{p^t_{b_i}}{}^{\mathbf{T}}\right]$$

$$\Delta B = \frac{1}{l}\sum_{i,j}^{l}\left[[\mathbf{p^t_{b_i}}\times]([\mathbf{\Delta p_{r_j}}\times] + [\mathbf{\Delta p_{b_j}}\times])\right]$$

$$\Delta B_n = \sum_{i=1}^{k}\left[[\mathbf{n^t_{b_i}}\times]([\mathbf{\Delta n_{b_i}}\times] - [\mathbf{\Delta n_{r_i}}\times]) + \mathbf{\Delta n_{r_i}}\mathbf{n^t_{b_i}}{}^{\mathbf{T}} + \mathbf{n^t_{b_i}}\mathbf{\Delta n_{b_i}}{}^{\mathbf{T}}\right]$$

$$\Delta B_p = \sum_{i=1}^{l}\left[[\mathbf{p^t_{b_i}}\times]([\mathbf{\Delta p_{b_i}}\times] - [\mathbf{\Delta p_{r_i}}\times]) + \mathbf{\Delta p_{r_i}}\mathbf{p^t_{b_i}}{}^{\mathbf{T}} + \mathbf{p^t_{b_i}}\mathbf{\Delta p_{b_i}}{}^{\mathbf{T}}\right]$$

$$\Delta Z = \frac{1}{l}\sum_{i,j}^{l}\left[[\mathbf{p^t_{b_i}}\times](\mathbf{\Delta p_{r_j}} - \mathbf{\Delta p_{b_j}})\right] + \sum_{i=1}^{k}\left[[\mathbf{n^t_{b_i}}\times](\mathbf{\Delta n_{b_i}} - \mathbf{\Delta n_{r_i}})\right] + \sum_{i=1}^{l}\left[[\mathbf{p^t_{b_i}}\times](\mathbf{\Delta p_{b_i}} - \mathbf{\Delta p_{r_i}})\right]$$

Inserting $A_0$(Eq. 18) and $\Delta A$(Eq. 19) into $A\boldsymbol{q} = \lambda_{max}\boldsymbol{q}$ yields:

$$(A_0 + \Delta A)\left(\begin{bmatrix}\mathbf{0_3} \\ 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\boldsymbol{\delta\theta} \\ 0\end{bmatrix}\right) = (\lambda_0 + \Delta\lambda)\left(\begin{bmatrix}\mathbf{0_3} \\ 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\boldsymbol{\delta\theta} \\ 0\end{bmatrix}\right) \tag{20}$$

After performing the multiplication, neglecting second-order terms, and canceling the common zeroth-order term $A_0 I_q = \lambda_0 I_q$ on both sides, the Eq. 20 simplifies to:

$$\frac{1}{2} A_0 \begin{bmatrix} \boldsymbol{\delta\theta} \\ 0 \end{bmatrix} + \Delta A \begin{bmatrix} \mathbf{0_3} \\ 1 \end{bmatrix} = \frac{1}{2} \lambda_0 \begin{bmatrix} \boldsymbol{\delta\theta} \\ 0 \end{bmatrix} + \Delta\lambda \begin{bmatrix} \mathbf{0_3} \\ 1 \end{bmatrix} \tag{21}$$

Inserting the explicit forms of $A_0$ and $\Delta B$ yields:

$$\begin{bmatrix} (B_{0n} + B_{0p} - 2B_0)\boldsymbol{\delta\theta} + 2\Delta Z^T \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\lambda_0 \boldsymbol{\delta\theta} \\ \Delta\lambda \end{bmatrix} \tag{22}$$

The fourth component of the equation confirms that the term is zero to first order with respect to vector errors. The first three components are crucial, as they provide the formula for the attitude error vector:

$$\boldsymbol{\delta\theta} = 2 \left[ \frac{1}{2}\lambda_0 I_3 - (B_{0n} + B_{0p} - 2B_0) \right]^{-1} \Delta Z \tag{23}$$

### *3.1.2 Translation Error Vector*

We can now solve for the dual part $\boldsymbol{q_d}$ as a function of the attitude quaternion $\boldsymbol{q}$:

$$\boldsymbol{q_d} = -(C_2 + C_2{}^T)^{-1} C_3{}^T \boldsymbol{q} \tag{24}$$

where $C_3 = C_3^0 + \Delta C_3$. The relationship between the translation vector and the dual quaternion is:

$$\boldsymbol{t} = \frac{1}{2} \begin{bmatrix} \mathbf{\Delta t} \\ 0 \end{bmatrix} = W(\boldsymbol{q})^T \boldsymbol{q_d} \tag{25}$$

From this, we determine the position error vector formula:

$$\mathbf{\Delta t} = \frac{2}{l} \left[ \sum_{i=1}^{l} [\mathbf{p_{b_i}^t} \times] \boldsymbol{\delta\theta} - \sum_{i=1}^{l} \left( \mathbf{\Delta p_{r_i}} - \mathbf{\Delta p_{b_i}} \right) \right] \tag{26}$$

## 3.2 Covariance Analysis

The vector error in the spacecraft's body and reference frames is typically unknown. Therefore, we conduct covariance analysis using statistical distributions with known expected values, assuming that these vector errors have zero mean:

$$E\{\mathbf{\Delta n_{r_i}}\} = \mathbf{E}\{\mathbf{\Delta n_{b_i}}\} = \mathbf{E}\{\mathbf{\Delta p_{r_i}}\} = \mathbf{E}\{\mathbf{\Delta p_{b_i}}\} = \mathbf{0} \tag{27}$$

Typically, the rationale behind this assumption is that any non-zero means of these quantities would be estimated and corrected prior to attitude estimation. In Eq. (23), the only random variables are represented by $\mathbf{\Delta n_{r_i}}, \mathbf{\Delta n_{b_i}}, \mathbf{\Delta p_{r_i}}, \mathbf{\Delta p_{b_i}}$, with all other quantities being deterministic. Since calculating expected values is a linear process, and the cross product of a vector with itself results in zero, it logically follows that:

$$E\{\boldsymbol{\delta\theta}\} = 2 \left[ \frac{1}{2}\lambda_0 I_3 - (B_{0n} + B_{0p} - 2B_0) \right]^{-1} E\{\Delta Z\} = \mathbf{0_3} \tag{28}$$

Furthermore, we make certain assumptions about the covariance of the vector errors. Assuming that the errors in each vector are uncorrelated, which implies that:

$$E\{\Delta\mathbf{n}_{\mathbf{b_i}}\Delta\mathbf{n}_{\mathbf{b_j}}^{\mathbf{T}}\} = \mathbf{E}\{\Delta\mathbf{n}_{\mathbf{r_i}}\Delta\mathbf{n}_{\mathbf{r_j}}^{\mathbf{T}}\} = \mathbf{E}\{\Delta\mathbf{p}_{\mathbf{b_i}}\Delta\mathbf{p}_{\mathbf{b_j}}^{\mathbf{T}}\} = \mathbf{E}\{\Delta\mathbf{p}_{\mathbf{r_i}}\Delta\mathbf{p}_{\mathbf{r_j}}^{\mathbf{T}}\} = \mathbf{0} \text{ for } \mathbf{i} \neq \mathbf{j} \tag{29}$$

$$E\{\Delta\mathbf{n}_{\mathbf{b_i}}\Delta\mathbf{n}_{\mathbf{r_i}}^{\mathbf{T}}\} = \mathbf{E}\{\Delta\mathbf{p}_{\mathbf{b_i}}\Delta\mathbf{p}_{\mathbf{r_i}}^{\mathbf{T}}\} = \mathbf{E}\{\Delta\mathbf{n}_{\mathbf{b_i}}\Delta\mathbf{p}_{\mathbf{r_i}}^{\mathbf{T}}\} = \mathbf{E}\{\Delta\mathbf{n}_{\mathbf{r_i}}\Delta\mathbf{p}_{\mathbf{b_i}}^{\mathbf{T}}\} = \mathbf{0} \text{ for all } \mathbf{i}, \mathbf{j} \tag{30}$$

The measurement covariance matrices are defined as follows:

$$\mathrm{R}_{\Delta\mathbf{n}_{\mathbf{b_i}}} = \mathrm{E}\{\Delta\mathbf{n}_{\mathbf{b_i}}\Delta\mathbf{n}_{\mathbf{b_i}}^{\mathbf{T}}\} \tag{31}$$

$$\mathrm{R}_{\Delta\mathbf{n}_{\mathbf{r_i}}} = \mathrm{E}\{\Delta\mathbf{n}_{\mathbf{r_i}}\Delta\mathbf{n}_{\mathbf{r_i}}^{\mathbf{T}}\} \tag{32}$$

$$\mathrm{R}_{\Delta\mathbf{p}_{\mathbf{b_i}}} = \mathrm{E}\{\Delta\mathbf{p}_{\mathbf{b_i}}\Delta\mathbf{p}_{\mathbf{b_i}}^{\mathbf{T}}\} \tag{33}$$

$$\mathrm{R}_{\Delta\mathbf{p}_{\mathbf{r_i}}} = \mathrm{E}\{\Delta\mathbf{p}_{\mathbf{r_i}}\Delta\mathbf{p}_{\mathbf{r_i}}^{\mathbf{T}}\} \tag{34}$$

Our focus is on the covariance matrix in terms of the rotation error and translation error:

$$P_{\delta\theta\delta\theta} = E\{(\boldsymbol{\delta\theta} - E\{\boldsymbol{\delta\theta}\})(\boldsymbol{\delta\theta} - E\{\boldsymbol{\delta\theta}\})^T\} = E\{\boldsymbol{\delta\theta}\boldsymbol{\delta\theta}^T\} \tag{35}$$

$$P_{\Delta t\Delta t} = E\{(\boldsymbol{\Delta t} - E\{\boldsymbol{\Delta t}\})(\boldsymbol{\Delta t} - E\{\boldsymbol{\Delta t}\})^T\} = E\{\boldsymbol{\Delta t}\boldsymbol{\Delta t}^T\} \tag{36}$$

Here, we give a special case under the assumption that the measurement vector errors and point errors have the same standard deviation between the reference frame and body frame, and the corresponding covariance matrix is diagonal, which allows us to separate the effects of noise and geometry.

$$\mathbf{R}_{\Delta\mathrm{n}_{\mathbf{b_i}}} = \mathbf{R}_{\Delta\mathrm{n}_{\mathbf{r_i}}} = \sigma_{\mathrm{n}}^{\mathbf{2}}\mathbf{I_3} \tag{37}$$

$$\mathbf{R}_{\Delta\mathrm{p}_{\mathbf{b_i}}} = \mathbf{R}_{\Delta\mathrm{p}_{\mathbf{r_i}}} = \sigma_{\mathrm{p}}^{\mathbf{2}}\mathbf{I_3} \tag{38}$$

The covariance expressions are as follows:

$$\begin{aligned} P_{\boldsymbol{\delta\theta}} = B(2\sigma_{\mathrm{n}}^2 \sum_{i=1}^{k} \alpha_i{}^2[\mathbf{n}_{\mathbf{b_i}}^{\boldsymbol{t}}\times][\mathbf{n}_{\mathbf{b_i}}^{\boldsymbol{t}}\times]^T + 2\sigma_{\mathrm{p}}^2 \sum_{i=1}^{l} \beta_i{}^2[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times][\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times]^T \\ + \frac{4\sigma_{\mathrm{p}}^2}{\beta} \sum_{i,j=1}^{l} \beta_i{}^2\beta_j[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times][\mathbf{p}_{\mathbf{b_j}}^{\boldsymbol{t}}\times]^T + \frac{2\sigma_{\mathrm{p}}^2}{\beta^2} \sum_{i,j,m=1}^{l} \beta_i{}^2\beta_j{}^2[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times][\mathbf{p}_{\mathbf{b_m}}^{\boldsymbol{t}}\times]^T)B \end{aligned} \tag{39}$$

$$\begin{aligned} P_{\mathbf{\Delta t}} = \frac{4}{\beta^2}[\sum_{i=1}^{l} \beta_i{}^2[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times]\, P_{\boldsymbol{\delta\theta}}[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times]^T + 2\sigma_{\mathrm{p}}^2 \sum_{i=1}^{l} \beta_i{}^2 I_3 + \frac{4\sigma_{\mathrm{p}}^2}{\beta} \sum_{i,j=1}^{l} \beta_i{}^2\beta_j[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times][\mathbf{p}_{\mathbf{b_j}}^{\boldsymbol{t}}\times]^T \\ + 2\sigma_{\mathrm{p}}^2((\sum_{i=1}^{l} \beta_i[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times])\, B(\sum_{j=1}^{l} \beta_j{}^2[\mathbf{p}_{\mathbf{b_j}}^{\boldsymbol{t}}\times])) + 2\sigma_{\mathrm{p}}^2((\sum_{i=1}^{l} \beta_i[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times])\, B(\sum_{j=1}^{l} \beta_j{}^2[\mathbf{p}_{\mathbf{b_j}}^{\boldsymbol{t}}\times]))^T \\ - \frac{2\sigma_{\mathrm{p}}^2}{\beta}((\sum_{i=1}^{l} \beta_i[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times])\, B(\sum_{i,j=1}^{l} \beta_i\beta_j{}^2[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times])) - \frac{2\sigma_{\mathrm{p}}^2}{\beta}((\sum_{i=1}^{l} \beta_i[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times])\, B(\sum_{i,j=1}^{l} \beta_i\beta_j{}^2[\mathbf{p}_{\mathbf{b_i}}^{\boldsymbol{t}}\times]))^T] \end{aligned} \tag{40}$$

where $B = 2[\frac{1}{2}\lambda_0 I_3 - (B_{0n} + B_{0p} - 2B_0)]^{-1}$. Eqs. (39) and (40) constitute the main results of this work. It is the first time that expressions for the covariance matrices of the estimation errors of this dual quaternion estimator are provided. The availability of these matrices means that the practitioners can assess the accuracy of the pose estimates, which is arguably as important as the estimated values themselves.

# 4 Simulation

This section validates the consistency of rotation and translation error, and their covariance through Monte Carlo simulations in specific scenarios. It also evaluates the algorithm's performance across varying measurement numbers.

## 4.1 Statistical Consistency

To verify the consistency of the predicted covariance, specific values for the attitude quaternion, translation vector, and measurements are set as follows: $\boldsymbol{q}_{RB} = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$, $\mathbf{t}_{RB} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, $\mathbf{n}_{\mathbf{r}_1} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $\mathbf{n}_{\mathbf{r}_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}$, $\mathbf{p}_{\mathbf{r}_1} = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$, $\mathbf{p}_{\mathbf{r}_2} = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$. A sample of 100,000 runs was created, and the Monte Carlo values were calculated using the standard sample average. Unit vectors and points in the reference coordinate system are transformed into the body coordinate system. To simulate measurement uncertainty, independent Gaussian noise for the reference and body frame measurements is introduced: $\sigma_\mathrm{n} = 0.01$ radians standard deviation along each axis for vectors, and $\sigma_\mathrm{p} = 0.01$ meters per axis for position points. The corresponding covariance matrices are diagonal.
Tables 2, 3 summarize the covariance matrices results for the attitude and translation errors. The results show that the formula predictions for both attitude error and translation error are on the same order of magnitude as the Monte Carlo results, approximately $\sigma^2$. The magnitudes and directions implied by the eigenvalues between the theoretical predictions and the Monte Carlo covariance matrix results match well, validating the accuracy of the theoretical model in predicting how the errors behave in real-world scenarios.

**Table 2 Covariance Results of Attitude Error Vector**

| ${P_{\delta\theta}}^{MC}(\times 10^{-4})$ | $P_{\delta\theta}(\times 10^{-4})$ | eigenvalue of ${P_{\delta\theta}}^{MC}, P_{\delta\theta}(\times 10^{-4})$ |
|---|---|---|
| $\begin{bmatrix} 1.0851 & -0.3956 & -0.5181 \\ -0.3956 & 1.1897 & 0.3973 \\ -0.5181 & 0.3973 & 1.1897 \end{bmatrix}$ | $\begin{bmatrix} 4.5730 & -2.1600 & -3.2669 \\ -2.1600 & 5.6800 & 2.1600 \\ -3.2669 & 2.1600 & 4.5730 \end{bmatrix}$ | $\begin{bmatrix} 0.5691, 1.3061 \\ 0.7995, 3.5200 \\ 1.9956, 10.0000 \end{bmatrix}$ |

**Table 3 Covariance Results for the Translation Error Vector**

| ${P_{\Delta\mathrm{t}}}^{MC}(\times 10^{-4})$ | $P_{\Delta\mathrm{t}}(\times 10^{-4})$ | eigenvalue of ${P_{\Delta\mathrm{t}}}^{MC}, P_{\Delta\mathrm{t}}(\times 10^{-4})$ |
|---|---|---|
| $\begin{bmatrix} 4.8639 & 1.1168 & -0.8928 \\ 1.1168 & 6.8991 & -1.1359 \\ -0.8928 & -1.1359 & 4.8798 \end{bmatrix}$ | $\begin{bmatrix} 4.1934 & 3.1869 & 4.52653 \\ 3.1869 & 2.8538 & 4.8130 \\ 4.52653 & 4.8130 & 4.1934 \end{bmatrix}$ | $\begin{bmatrix} 3.9790, 1.4611 \\ 4.6409, 0.5143 \\ 8.0229, 12.1876 \end{bmatrix}$ |

The analytical formulas for $\boldsymbol{P}_{\boldsymbol{\delta\theta}}$ and $\boldsymbol{P}_{\boldsymbol{\Delta\mathrm{t}}}$ enable the direct calculation of the error covariance in a single step. In contrast, MC-based sensitivity analysis requires a large ensemble of trials (e.g., $10^5$ runs in this study), which is computationally prohibitive for onboard flight computers. While MC methods serve as a numerical benchmark, our analysis shows that the analytical first-order approximations yield results on the same order of magnitude. This confirms that for standard sensing noise levels, the analytical method captures the system's stochastic behavior with high fidelity. The availability of closed-form covariance matrices enables the practitioner to assess the integrity of each pose estimate in real time.

# 5 Conclusion

This paper presents a comprehensive error analysis of a single-frame dual-quaternion batch estimator for simultaneous attitude and position estimation. The main contribution is the development of first-order analytical expressions for the attitude and translation error vectors. And the corresponding covariance matrices. Extensive Monte Carlo simulations validated the accuracy of the derived analytical expressions, demonstrating the agreement between theoretical predictions and empirical results. These covariance expressions enable practitioners to analytically assess pose estimation accuracy, which is essential for real-time applications where Monte Carlo analysis is impractical. Future work will extend this analysis to sequential estimation frameworks and investigate robustness under non-Gaussian measurement noise.

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